

ON QUANTUM GROUPS CO-REPRESENTATIONS

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Abstract

We carry out a generalization of quantum group co-representations in order to encode in this structure those cases where non-commutativity between endomorphism matrix entries and quantum space coordinates happens.

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1 Introduction

Quantum Groups arise as the abstract structure underlying the symmetries of integrable systems in (1+1) dimensions[1]. There, the theory of quantum inverse scattering give rise to some deformed algebraic structures which were first explained by Drinfeld as deformations of classical Lie algebras[2][3]. Also, an analog structure was obtained by Woronowicz in the context of non-commutative C^* -algebras[4]. There is a third approach, due to Manin, where Quantum Groups are interpreted as the endomorphisms of certain non commutative algebraic varieties defined by quadratic algebras, called quantum linear spaces (QLS)[5][6]. In this work, we are mainly involved with this last approach.

From the algebraic point of view, quantum groups are Hopf algebras and the relation with the endomorphism algebra of QLS come from their co-representations on tensor product spaces. It is worth remarking that the usual construction of the co-action on the tensor product space implies the commutativity between the matrix elements of a representation of the endomorphism and the coordinates of the QLS.

In the present work, we are interested in relaxing this last statement thus admitting non-commutative relations between endomorphism matrix entries and quantum linear space coordinates. We carry out this task modifying the definition of co-action on tensor product spaces. The question is: at what extent can this be done without spoling out the bialgebra and its co-representation structure? We answer this question, finding the conditions under which the general framework of quantum groups and quantum linear space still holds. This construction introduces a new family of deformation parameters in the endomorphism bialgebra of the QLS, as it is shown in the quantum plane example where a two parameter $M_{q,p}(2)$ is obtained. Also, we find a non central object playing the rol of (q,p) -deformed determinant which allows to obtain a $GL_{q,p}(2)$.

We present a brief description of the co-representations of bialgebras in the second section, developing our approach to modified co-representations in the third one and, finally, we present the quantum plane example in the last section.

2 Quantum algebras and co-representations

Let V be a vector space of dimension n , $\{e_i\}$ a basis for V and H_o the trivial bialgebra of functions over $GL(n, \mathbf{C})$. This bialgebra is freely generated by the identity and the coordinates functions T_i^j , in the basis $\{e_i\}$, defined by

$$T_i^j : GL(n, \mathbf{C}) \longrightarrow \mathbf{C}$$

$$T_i^j : g \longmapsto g_i^j$$

for $g \in GL(n, \mathbf{C})$.

The co-product and co-unit are given by

$$\Delta T_i^j = T_i^k \otimes T_k^j \quad (1)$$

$$\varepsilon(T_i^j) = \delta_j^i \quad (2)$$

From now on, sumation over repeated index is assumed.

The comodule (δ, V) , with

$$\delta : V \longrightarrow H_o \otimes V$$

$$\delta(e_i) = T_i^j \otimes e_j \quad (3)$$

provides a representation of $GL(n, \mathbf{C})$ in V , through the g_i^j in the basis b of V . It has the co-associativity property and preserves the co-unit, which is expressed by the commutativity of the diagrams

$$\begin{array}{ccc} V & \xrightarrow{\delta_V} & H_o \otimes V \\ \delta_V \downarrow & & \downarrow I_{H_o} \otimes \delta_V \\ H_o \otimes V & \xrightarrow{\Delta \otimes I_V} & H_o \otimes H_o \otimes V \end{array} \quad (4)$$

$$\begin{array}{ccc} & H_o \otimes V & \\ \delta_V \nearrow & & \searrow \epsilon \otimes I_V \\ V & \xrightarrow{I_V} & V \end{array} \quad (5)$$

Building up co-representations for objects with more structure than V^\otimes , as quadratic algebras for example, requires some extra conditions that we sketch below.

Let A denote the quadratic algebra generated by the ideal $I(\mathfrak{B})$, where $\mathfrak{B} : V \otimes V \longrightarrow V \otimes V$, then

$$A(\mathfrak{B}) = \frac{V^\otimes}{I(\mathfrak{B})} \quad (6)$$

V^\otimes is tensor algebra on V . In general we consider \mathfrak{B} with the form

$$\mathfrak{B} = (I_{V \otimes V} - B) \quad (7)$$

$$e_i e_j - B_{ij}^{kl} e_k e_l \quad (8)$$

In order to built up a co-module structure on $A(\mathfrak{B})$, one needs to work on the tensor algebra V^\otimes . This is obtained with the natural extension of the co-action on $V \otimes V$:

$$\delta_{V \otimes V} = (m \otimes I_V)(I_{H_o} \otimes \tau \otimes I_V)(\delta_V \otimes \delta_V) \quad (9)$$

where $m : H_o \otimes H_o \longrightarrow H_o$ is the product in the bialgebra H_o . It is worth remarking that the flip operator $\tau : V \otimes H_o \longrightarrow H_o \otimes V$ will render a commutative product between the T_i^j and e_k , as it is assumed in the construction of the quantum plane and $GL_q(2)$ [5]

The main condition for preserving the quadratic algebra structure is the "commutativity" between \mathfrak{B} and $\delta_{V \otimes V}$:

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\mathfrak{B}} & V \otimes V \\ \delta_{V \otimes V} \downarrow & & \downarrow \delta_{V \otimes V} \\ H \otimes V \otimes V & \xrightarrow{I_H \otimes \mathfrak{B}} & H \otimes V \otimes V \end{array} \quad (10)$$

This graph is satisfied if H is the bialgebra arising from the quotient of the free algebra generated by the objects T_i^j and the ideal $I(\mathfrak{B}, H_o)$ generated by

$$B_{ij}^{kl} T_k^r T_l^s - T_i^k T_j^l B_{kl}^{rs} \quad (11)$$

i.e.,

$$H = \frac{H_o}{I(\mathfrak{B}, H_o)}$$

Since $I(\mathfrak{B}, H_o)$ is a co-ideal with relation to Δ then H becomes a bialgebra. The equation (11) is a central object in the so called FRT construction[7]. In this way, $A(\mathfrak{B})$ becomes in a H -algebra comodule.

3 Generalized co-representations

The main aim of this section is to build up the mathematical framework encoding the situation in which entries of the endomorphism matrix may not commute with the coordinates of the quantum linear space defined in (18). We will reach it by means of a modification in the co-representation theory. As described in the previous section, supplying the quantum linear space with a H -comodule structure requires a good definition of a co-action on $V \otimes V$.

As we saw, the standard definition of $\delta_{V \otimes V}$, eq.(9), provides both $V \otimes V$ with a H_o -comodule structure and $A = \frac{V \otimes}{I(\mathfrak{B})}$ with an H -comodule structure.

We will show that one can replace the flip map τ by a more general one γ in such a way that there exist a H_γ -comodule structure, for some H_γ to be constructed, provided γ satisfy some requirements. Then, let γ be defined by

$$\gamma : V \otimes H \longrightarrow H \otimes V \quad (12)$$

$$\gamma(e_i T_j^k) = \gamma_{ijn}^{klm} T_l^m e_m \quad (13)$$

So, we propose the co-action on tensor product space as being

$$\delta_{V \otimes V}^\gamma = (m \otimes I_V)(I_{H_\gamma} \otimes \gamma \otimes I_V)(\delta_V \otimes \delta_V) \quad (14)$$

In order to have an H_γ -comodule structure, the co-associativity and co-unity properties, diagrams (4) and (5), must be satisfied together with requirement that \mathfrak{B} be a co-module map, (10).

Recalling the bijection between comodules and multiplicative matrices[5], let us consider the multiplicative matrix M in $V \otimes V$ with coefficients in H_o corresponding to the comodule $\delta_{V \otimes V}^\gamma$, i.e.,

$$\delta_{V \otimes V}^\gamma \equiv M \in \text{End}(V \otimes V, H_o)$$

$$\delta_{V \otimes V}^\gamma(e_i \otimes e_j) = M_{ij}^{rs} \otimes e_r \otimes e_s \quad (15)$$

hence M is

$$M_{ij}^{kl} = \gamma_{mjn}^{lpk} T_i^m T_p^n. \quad (16)$$

Also, we adopt the following convention: let A_{ij}^{kl} and D_{ij}^{kl} any pair of four-tensors, then we write

$$(A \times B)_{ij}^{rs} = A_{ij}^{kl} \times D_{kl}^{rs} \quad (17)$$

where \times stands for any kind of product (tensor, algebraic, etc.), and sum over repeated index is assumed.

We start analyzing the diagram (10), which means that the co-module $\delta_{V \otimes V}^\gamma$ map preserves the quantum linear space $A(\mathfrak{B})$

$$A(\mathfrak{B}) = \frac{V^\otimes}{I(\mathfrak{B})}, \quad (18)$$

that it is expressed by the commutation relation

$$(I_{H_\gamma} \otimes \mathfrak{B}) \circ \delta_{V \otimes V}^\gamma = \delta_{V \otimes V}^\gamma \circ \mathfrak{B}$$

From this equation we get the condition

$$\mathfrak{B}M = M\mathfrak{B}$$

meaning that H_γ must be defined as being

$$H_\gamma = \frac{H_o}{I(\mathfrak{B}M - M\mathfrak{B})}$$

Here $I(\mathfrak{B}M - M\mathfrak{B})$ is the ideal generated by the quadratic relation $\mathfrak{B}M - M\mathfrak{B}$.

A necessary and sufficient condition for H_γ to be a bialgebra is that $I(\mathfrak{B}M - M\mathfrak{B})$ be a co-ideal, i.e.,

$$\Delta I \subset I \otimes H + H \otimes I \quad (19)$$

Here, it must be taken into account that in order to have a H_γ -comodule structure on $A(\mathfrak{B})$, $\delta_{V \otimes V}^\gamma$ must fulfill the co-associativity and co-unity properties, diagrams (4) and (5),

$$(\Delta \otimes I_{V \otimes V}) \circ \delta_{V \otimes V}^\gamma = (I_{H_\gamma} \otimes \delta_{V \otimes V}^\gamma) \circ \delta_{V \otimes V}^\gamma$$

$$(\epsilon \otimes I_{V \otimes V}) \circ \delta_{V \otimes V}^\gamma = I_{V \otimes V},$$

Again, because of the bijection between all the structures of left co-module on $V = \mathbf{C}^n$ and the multiplicative matrix $\mathbf{M}(n, H_\gamma)[5]$, if $M \in H_\gamma$ satisfy

$$\Delta M = M \otimes M \quad (20)$$

$$\epsilon(M) = I \quad (21)$$

automatically both properties are satisfied.

Then, coming back to the co-ideal question, eq. (19), and assuming that this last condition holds, one gets

$$\begin{aligned} \Delta(\mathfrak{B}M - M\mathfrak{B}) &= \mathfrak{B}\Delta M - M\Delta\mathfrak{B} \\ &= \mathfrak{B}(M \otimes M) - (M \otimes M)\mathfrak{B} \\ &= (\mathfrak{B}M - M\mathfrak{B}) \otimes M - M \otimes (\mathfrak{B}M - M\mathfrak{B}) \end{aligned}$$

and

$$\epsilon(\mathfrak{B}M - M\mathfrak{B}) = \mathfrak{B}\epsilon(M) - \epsilon(M)\mathfrak{B} = 0$$

hence H_γ is a bialgebra.

This results may be resumed into the following assertion: if there exist γ satisfying $\mathfrak{B}M - M\mathfrak{B}$, $\Delta M = M \otimes M$ and $\epsilon(M) = I_{V \otimes V}$, for $M \in H_\gamma = \frac{H_\sigma}{I(\mathfrak{B}M - M\mathfrak{B})}$, then H_γ is a bialgebra and $\delta_{V \otimes V}^\gamma$ renders $A(\mathfrak{B})$ into a H_γ -comodule.

Now, we consider the antipode. If one can define an antipode, i.e., a map $S : H_\gamma \longrightarrow H_\gamma$ such that the following property holds

$$m \circ (S \otimes I_{H_\gamma}) \circ \Delta = m \circ (I_{H_\gamma} \otimes S) \circ \Delta = \eta \circ \epsilon \quad (22)$$

at the level of the multiplicative matrices representing the bialgebra H_γ in $A(\mathfrak{B})$ via a comodule $(A(\mathfrak{B}), \delta)$, then we can assert that there exists a representation of the resulting Hopf algebra, given by the comodule $(A(\mathfrak{B}) \otimes A(\mathfrak{B}), \delta_{V \otimes V}^\gamma)$. In fact recalling that,

$$S(M_{ij}^{kl}) = \gamma_{mjn}^{lpk} S(T_p^n) S(T_i^m). \quad (23)$$

it is simple to see that,

$$m \circ (S \otimes I_{H_\gamma}) \circ \Delta M = m \circ (I_{H_\gamma} \otimes S) \circ \Delta M = \eta \circ \epsilon(M)$$

and using (20) ensures that $S(M)$ is the inverse of M ,

$$MS(M) = S(M)M = I$$

So, we may affirm that the construction presented above still holds if H_γ is a Hopf algebra, then encoding the situation of non-commutative co-representations in the framework of Quantum Groups.

In the next section, we describe an explicit example enjoying all these properties presented above, namely a p -deformed version of the endomorphism of the quantum plane.

4 The Quantum Plane

Let the quantum plane $A_q^{2|0}$ described by

$$e_1 e_2 = q e_2 e_1 \quad (24)$$

In the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$, this relation can be expressed by means the quadratic form B as

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & 1 - q^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (25)$$

which is a solution of the Yang-Baxter equation

$$B_{12} B_{23} B_{12} = B_{23} B_{12} B_{23} \quad (26)$$

It is worth remarking that the following construction leads to the same structure for other choice of B , as the symmetric and idempotent B' ,

$$B' = \frac{1}{q + q^{-1}} \begin{bmatrix} q + q^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & 2 & 0 \\ 0 & 2 & q^{-1} - q & 0 \\ 0 & 0 & 0 & q + q^{-1} \end{bmatrix} \quad (27)$$

This B' is not a solution of Yang-Baxter equation but, in the Manin construction for pseudo-symmetric quantum space[5], it enables to characterize all the endomorphism of the quantum plane by the relation $B'M - MB'$ as the only solution to the master relation $(I - B')M(I + B')$.

The endomorphism matrix T is

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (28)$$

Then, proposing the simplest form for γ ,

$$\gamma_{ijn}^{klm} = \delta_i^m \delta_j^l \delta_n^k g(i, j, k) \quad (29)$$

we get M ,

$$M_{ij}^{kl} = g(k, j, l) T_i^k T_j^l \quad (30)$$

Thus, solving the pair of conditions

$$\Delta M = M \otimes M$$

$$BM = MB$$

we obtain the following $g(i, j, k)$

$$\begin{aligned} g(i, j, j) &= 1 \\ g(i, 1, 2) &= p \\ g(i, 2, 1) &= p^{-1} \end{aligned} \quad (31)$$

where $p \in \mathbf{C}$ is new parameter. Introducing this g in γ , and substituting in the relation $BM = MB$ one gets

$$\begin{aligned} ac - pqca &= 0 \\ ab - p^{-1}qba &= 0 \\ bc - p^2cb &= 0 \\ cd - p^{-1}qdc &= 0 \\ bd - pqdb &= 0 \\ ad - da + p(q^{-1} - q)cb &= 0 \end{aligned} \quad (32)$$

which define a two parametric $M_{q,p}(2)$ as the quotient algebra

$$M_{q,p}(2) = \frac{k[T_i^j]}{I(BM - MB)} \quad (33)$$

for i and j from 1 to 2.

Also, this g give rise to the following relations between matrix entries and the coordinates of the quantum plane,

$$\begin{aligned} e_i a - a e_i &= 0 \\ e_i d - d e_i &= 0 \\ e_i b - p b e_i &= 0 \end{aligned} \tag{34}$$

$$e_i c - p^{-1} c e_i = 0$$

The same construction can be carried out on the Grassmannian plane $A_q^{0|2}$ given by the quadratic relations

$$e_1 e_2 = -q^{-1} e_2 e_1 \tag{35}$$

$$e_i^2 = 0 \tag{36}$$

leading to same set of relations (32) and (34), so $M_{q,p}(2)$ is also the bialgebra of endomorphism of $A_q^{0|2}$.

It is also possible to define an antipode on $M_{q,p}(2)$. First at all, we need to define something like a determinant, but there is no such an object in the center of $M_{q,p}(2)$. However we may built up a $\det_{q,p}$ such that it has homogenous commutation relation with the generator of $M_{q,p}(2)$, i.e., $\{a, b, c, d\}$. Then, we define $D = \det_{q,p} \in \frac{k[a,b,c,d]}{I(BM-MB)}$,

$$D = \det_{q,p} = ad - p^{-1} q bc \tag{37}$$

that satisfy the following commutation relations

$$\begin{aligned} Da - aD &= 0 \\ Db - p^{-2} bD &= 0 \\ Dc - p^2 cD &= 0 \\ Dd - dD &= 0 \end{aligned} \tag{38}$$

With these properties, and assuming that D is an invertible element of $M_{q,p}(2)$ we define the antipode S

$$S \begin{bmatrix} a & b \\ c & d \end{bmatrix} = D^{-1} \begin{bmatrix} d & -(pq)^{-1}b \\ -pqc & a \end{bmatrix} \quad (39)$$

and, consequently, we have the Hopf algebra $GL_{q,p}(2)$.

5 Concluding Remarks

We have introduced a new ingredient in the theory of representations of Quantum (semi)Groups admitting non-commutativity between endomorphism matrix entries and quantum space coordinates. This feature give rise to an extra deformation of all the involved structures. In fact, we have shown that it is possible to introduce the map $\gamma : V \otimes H_\gamma \longrightarrow H_\gamma \otimes V$ without spoiling out the Hopf algebra and co-module structures provided that γ turns H_γ into a Quantum Matrix Group. This was explicitly shown in the Quantum Plane example, where we obtained a two parameter deformation $M_{q,p}(2)$ and also we were able to construct a H_γ valued function $\det_{q,p}$, out of the centre of the algebra, thus making possible to get $GL_{q,p}(2)$.

6 Acknowledgment

We are greatly indebted to M. L. Bruschi for enlightening discussions. Also, the authors thank CONICET-Argentina for financial support.

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